idea/goal: classify root systems $\longrightarrow$ classify semisimple lie algebras

first: pick "generating set" of R

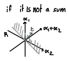
$\exists t \in E$ s.t. $\forall \alpha \in R \subset E$, $(t,\alpha) \neq 0$

t determines a polarization of R:

$R = \{(\alpha,t)>0\} \sqcup \{(\alpha,t)<0\}$

$\overset{\shortparallel}{R_+}$ positive roots

$\alpha \in R_+$ is simple if it is not a sum of positive roots, $\Pi = \{\text{simple roots}\}$

Example: 

$R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$

$\Pi = \{\alpha_1, \alpha_2\}$

Thrm: simple roots form a basis of E

Follows from: • all $\alpha \in R_+$ are sums of simple roots

• $\alpha, \beta \in \Pi$ then $(\alpha,\beta) \leqslant 0$

Cor: $\forall \alpha \in R$, $\alpha = \sum_{i=1}^{r} n_i \alpha_i$ with $\alpha_i \in \Pi$

n_i either all $\geqslant 0$ or $\leqslant 0$

root & coroot lattices: $Q \subset E$, $Q^\vee \subset E^*$ abelian grps

$Q = \langle \alpha \in R \rangle$, $Q^\vee = \langle \alpha^\vee \rangle$

with basis of simple roots: $Q = \bigoplus \mathbb{Z}\alpha_i$

$Q^\vee = \bigoplus \mathbb{Z}\alpha_i^\vee$

weight lattice: $P \subset E$, $P = \{\lambda : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha^\vee \in Q^\vee\}$

P is the dual of $Q^\vee$ & $P \subset Q$ though in general $\neq$

again, $P = \{\lambda : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \ \forall \alpha_i \in \Pi\}$

fundamental weight $\omega_i \in E$ s.t. $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$

$P = \bigoplus \mathbb{Z}\omega_i$

Example: • A_1 has unique α positive, $Q = \mathbb{Z}\alpha$, $Q^\vee = \mathbb{Z}\alpha^\vee$

with $(\alpha,\alpha) = 2$, $\langle \alpha, \alpha^\vee \rangle = 2$

then, $\omega = \frac{\alpha}{2} \Rightarrow P = \mathbb{Z}\frac{\alpha}{2} \Rightarrow P/Q = \mathbb{Z}_2$

Simple roots Π depend on choice of $t \in E$, is there some equivalence?

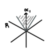
with $L_\alpha = \{\lambda \in E : (\alpha,\lambda) = 0\}$ hyperplane orthogonal to $\alpha \in R$

then, $t \in E \setminus \bigcup_{\alpha \in R} L_\alpha$

this splits into connected components C called Weyl chambers

Π fixed $\forall t \in C$

given a choice of $t \in E$ can define positive Weyl chamber



$C_+ = \{\lambda \in E : (\lambda, \alpha_i) > 0 \ \forall \alpha_i \in \Pi\}$

$\therefore$ bijection $\{\text{polarizations of } R\} \longleftrightarrow \{\text{Weyl chambers}\}$

Thrm: the Weyl grp W act transitively on Weyl chambers

& $\forall$ polarizations Π, Π', $\exists w \in W$ s.t. $\Pi = w(\Pi')$

Can we recover R from Π?

Thrm: R is a reduced system w/ polarization Π

1) reflections s_{α_i} generate W

2) $W(\Pi) = R$

Cor: R can be recovered from Π

Classifying $R \Leftrightarrow$ Classifying Π

a root system R is reducible if $\exists R_1, R_2$ orthogonal s.t. $R = R_1 \sqcup R_2$

Lemma: R reduced then R reducible $\Leftrightarrow \Pi = \Pi_1 \sqcup \Pi_2$ where Π

suffices to consider R irreducible

Cartan matrix: $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$

Lemma: • $a_{ii} = 2$

• $i \neq j$, $a_{ij} \in \mathbb{Z}$ & $a_{ij} < 0$

• $i \neq j$, $a_{ij}a_{ji} = 4\cos^2\varphi$ for $\varphi = \alpha_i \measuredangle \alpha_j$

if $\varphi \neq \frac{\pi}{2}$, $\frac{|\alpha_i|^2}{|\alpha_j|^2} = \frac{a_{ji}}{a_{ij}}$

Dynkin diagram of Π: • $\alpha_i \in \Pi \longrightarrow$ vertex v_i

• $\alpha_i \neq \alpha_j \longrightarrow$ connect by n edges, depending on $\varphi = \alpha_i \measuredangle \alpha_j$

φ	n
$\pi/2$	0
$2\pi/3$	1
$3\pi/4$	2
$5\pi/6$	3

• $|\alpha_i| \neq |\alpha_j|$ & not orthogonal $\longrightarrow$ orient edge towards shorter root

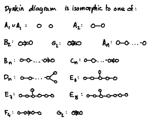
Thrm: 1) Dynkin diagram of Π connected $\Leftrightarrow$ R irreducible

2) Dynkin diagram determines Cartan matrix

3) R determined by Dynkin diagram uniquely up to isomorphism

Thrm: R reduced irreducible root system

it's Dynkin diagram is isomorphic to one of:



Classifying $R \longrightarrow$ Classifying $\mathfrak{g}$

Thrm: $\mathfrak{g}$ semisimple complex lie algebra w/ root system $R \subset \mathfrak{h}^*$

$(\,,\,)$ a non-degenerate, invariant, symmetric bilinear form on $\mathfrak{g}$

polarization w/ Π, then:

1) $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$ are subalgebras in $\mathfrak{g}$

and, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

2) with $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ s.t. $(e_i, f_i) = \frac{2}{(\alpha_i,\alpha_i)}$

and $h_i = \frac{2H_{\alpha_i}}{(\alpha_i,\alpha_i)}$ where $H_\alpha \in \mathfrak{h}$ corresponds to $\alpha \in \mathfrak{h}^*$

$\{e_1 \dots e_r\}$ generate $\mathfrak{n}_+$

$\{f_1 \dots f_r\}$ generate $\mathfrak{n}_-$

$\{h_1 \dots h_r\}$ basis of $\mathfrak{h}$

3) Serre relations: $[h_i, h_j] = 0$

$[h_i, e_j] = a_{ij}e_j$, $[h_i, f_j] = -a_{ij}f_j$

$[e_i, f_j] = \delta_{ij}h_i$

$(\mathrm{ad}_{e_i})^{1-a_{ij}}e_j = (\mathrm{ad}\,f_i)^{1-a_{ij}}f_j = 0$

where a_{ij} is the Cartan matrix

Thrm: for R reduced irr. root system & $\mathfrak{g}(R)$ complex lie algebra generated by e_i, f_i, h_i

then, $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra with root system R

Cor: 1) if $\mathfrak{g}$ has root system R, natural iso. $\mathfrak{g} \simeq \mathfrak{g}(R)$

2) bijection $\{\text{isom. classes of reduced root systems}\} \longleftrightarrow \{\text{isomorphism classes of finite dim. complex semisimple lie algebras}\}$

$\therefore$ these are clasified by the Dynkin diagrams